1. Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x, & 0 \leq x < 2 \end{cases}$$

$$f(x+4) = f(x)$$

Determine the coefficients in this Fourier series.

(2 marks)

2. Let $f(x) = \begin{cases} 0, & -3 < x < -1 \\ 1, & -1 < x < 1 \\ 0, & 1 < x < 3 \end{cases}$ and suppose that $f(x+6) = f(x)$, Determine whether $f(x)$ is odd or even function and express $f(x)$ in Fourier series form.

(3 marks)

3. Consider a vibrating string of length $L = 30$ that satisfies the wave equation

$$4\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2}, \quad 0<x<30, \qquad t>0$$

Assume that the ends of the string are fixed, and that the string is set in motion with no initial velocity from the initial position

$$u(x,0) = f(x) = \begin{cases} \dfrac{x}{10}, & 0 \leq x \leq 10 \\ \dfrac{(30-x)}{20}, & 10 < x \leq 30 \end{cases}$$

Find the displacement $u(x,t)$ of the string.

(2.5 marks)

4. Find the temperature $u(x,t)$at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature of 20ºC throughout, and whose ends are maintained at 0ºC for all t>0.

(2.5 marks)

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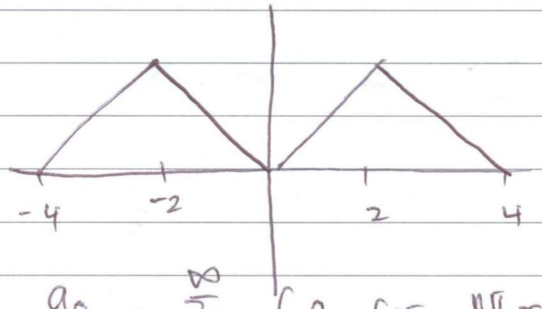
STUDENT ID (IN FIGURES) : ....

SUBJECT : ....

DATE : TEST - 2 solution

①
$$f(x) = \begin{cases} -x & \rightarrow \quad -2 \le x < 0 \\ x & \rightarrow \quad 0 \le x < 2 \end{cases}$$



$f(x)$ = Even func

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-2}^{0} -x\,dx + \frac{1}{2L}\int_{0}^{2} x\,dx$$

$$= \frac{1}{4}\int_{-2}^{0} -x\,dx + \frac{1}{4}\int_{0}^{2} x\,dx$$

$$a_0 = \frac{2}{2} = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[-\frac{2}{n\pi}x\sin\frac{n\pi x}{2} - \left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right|_{-2}^{0} + \frac{1}{2}\left[\frac{2}{n\pi}x\sin\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right]_0^2$$

$$a_n = \left(\frac{4}{(n\pi)^2}\right)(\cos n\pi - 1)$$

$$a_n \begin{cases} \dfrac{-8}{(n\pi)^2} & \rightarrow \quad n = \text{odd} \\ 0 & \rightarrow \quad n = \text{even} \end{cases}$$

$$b_n = 0$$

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}$$

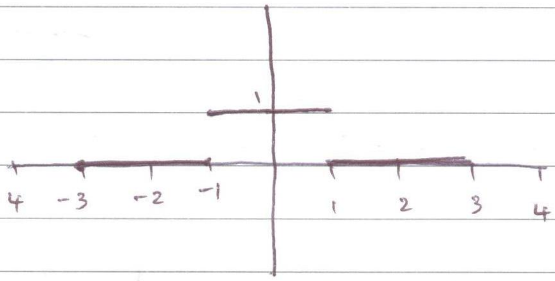
②

$$f(x) = \begin{cases} 0, & \rightarrow \quad -3 < x < -1 \\ 1, & \rightarrow \quad -1 < x < 1 \\ 0, & \rightarrow \quad 1 < x < 3 \end{cases}$$

$f(x+6) = x$

$2L = 6$

$L = 3$



Even func

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi}{3} x \right)$$

$$a_0 = \frac{1}{2.3} \int_{-3}^{3} f(x)\,dx = \frac{1}{6} \int_{-1}^{1} dx = \frac{1}{3}$$

$$a_n = \frac{1}{3} \int_{-1}^{1} \cos \frac{n\pi x}{3}\,dx = \frac{1}{n\pi} \sin \frac{n\pi x}{3} \Big|_{-1}^{1} = \frac{2}{n\pi} \sin \frac{n\pi}{3} \Rightarrow n = 1, 2, 3, \ldots$$

$b_n = 0$

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{3} \cos \frac{n\pi x}{2}$$

$$f(x) = \frac{1}{3} + \frac{\sqrt{3}}{\pi} \left[\cos\left(\frac{\pi x}{3}\right) + \frac{\cos\left(\frac{2\pi x}{3}\right)}{2} - \frac{\cos(4\pi x/3)}{4} - \frac{\cos(5\pi x/3)}{5} + \cdots \right]$$

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③ $L = 30$

$$A \frac{d^2u}{dx^2} = \frac{d^2u}{dt^2} \quad ; A = 4$$

$c^2 = A \rightarrow c = 2$

$$u(x,0) = f(x) = \begin{cases} \frac{x}{10} & \rightarrow 0 \le x \le 10 \\ \frac{30-x}{20} & \rightarrow 10 \le x \le 30 \end{cases} \Bigg\} \text{ velocity} = 0 \rightarrow D_n = 0$$

$u(x,t) = \ldots ?$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{30} \cos \frac{2n\pi t}{30}$$

$$C_n = \frac{2}{30} \int_0^{10} \frac{x}{10} \sin \frac{n\pi x}{30}\,dx + \frac{2}{30} \int_{10}^{30} \frac{30-x}{20} \sin \frac{n\pi x}{30}\,dx \leftarrow$$

Solve using integration by part.

$$C_n = \frac{9}{n^2\pi^2} \sin \frac{n\pi}{3} \quad , \; n = 1, 2, 3 \ldots$$

to be continve

③ continue . . .

$$u(x,t) = \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{30} \cos\frac{2n\pi t}{30}$$

$$u(x,t) = \frac{9}{\pi^2} \sin\frac{\pi}{3} \sin\frac{\pi x}{30} \cos\frac{2\pi t}{30} + \frac{9}{4\pi^2} \sin\frac{2\pi}{3} \sin\frac{2\pi x}{30} \cos\frac{4\pi t}{30} + 0 + \dots$$

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④ $L = 50$

$f(x) = 20 \quad \longrightarrow \quad 0 < x < 50$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{2} \, dx$$

$$= \frac{2}{50}\int_0^{50} 20 \cdot \sin\frac{n\pi x}{50} \, dx$$

$$B_n = \frac{40}{n\pi}\left(-\cos n\pi + 1\right) \begin{cases} \frac{80}{n\pi} & \rightarrow n = \text{odd} \\ 0 & \rightarrow n = \text{even.} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{80}{n\pi} \sin\frac{n\pi x}{50} e^{-\left(\frac{c^2 n^2 \pi^2 t}{2500}\right)}$$

$$u(x,t) = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{50} e^{-\left(\frac{c^2 n^2 \pi^2 t}{2500}\right)}$$

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